PERGAMON

Applied Mathematics Letters 16 (2003) 415–420

Applied Mathematics Letters

www.elsevier.com/locate/aml

Orthogonality in Multiobjective Optimization

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(Received and accepted February 2002)

Abstract—Properties of nonlinear multiobjective problems implied by the Karush-Kuhn-Tucker necessary conditions are investigated. It is shown that trajectories of Lagrange multipliers corresponding to the components of the vector cost function are orthogonal to the corresponding trajectories of vector deviations in the balance space (to the balance set for Pareto solutions).

Keywords—Vector optimization, Nonscalarized multiobjective programming, Balance set.

1. INTRODUCTION

Let k, q, m, and n be integers, Ω and P closed subsets of $\mathbb{R}^n$ and $\mathbb{R}^q$, respectively, $l = (l_1, l_2, \dots, l_k) \in \mathbb{R}^k_+$ with $l_i \geq 0$ for $i = 1, 2, \dots, k$ and $l \neq 0$, and $f : \Omega \times P \longrightarrow \mathbb{R}^k$ and $g : \Omega \times P \longrightarrow \mathbb{R}^m$ two vector functions whose real components will be denoted by f_i, $i = 1, 2, \dots, k$, and g_j, $j = 1, 2, \dots, m$, respectively. Consider the following vector optimization problem:

$$\operatorname{Min} f(x,p) \quad \begin{cases} x \in \Omega, \\ g(x,p) \leq 0, \end{cases} \tag{1}$$

where $p \in P$ is a fixed vector-parameter.

ASSUMPTION A1. *Suppose, that the scalar problem*

$$\operatorname{Min}\{f_i(x,p) : x \in \Omega,\ g(x,p) \leq 0\} \tag{2}$$

attains its optimal value at a single point $x(i,p) \in \Omega$ $(i = 1, 2, \dots, k)$ and, therefore, the ideal point (or the set of partial minima)

$$J(p) = [f_1(x(1,p)), f_2(x(2,p)), \dots, f_k(x(k,p))] \in \mathbb{R}^k$$

does exist. To simplify notations, we will denote $J_i(p) = f_i(x(i,p))$, $i = 1, 2, \dots, k$.

0893-9659/03/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0893-9659(02)00214-8

Following the approach of [1] or [2], an element $b \in \mathbb{R}^k$, $b \geq 0$, is said to be a *balance point* of (1) if $\{f(x) : x \in \Omega,\ g(x,p) \leq 0\} \cap [J(p), J(p)+b] \neq \emptyset$ and $\{f(x) : x \in \Omega,\ g(x,p) \leq 0\} \cap [J(p), J(p)+b^*] = \emptyset$ for every $b^* \in \mathbb{R}^k$ such that $0 \leq b^* \leq b$, $b^* \neq b$.[1] As pointed out in [1], $b \in \mathbb{R}^k$ is a balance point of (1) if and only if $J(p)+b$ is the value $f(x_p)$ at a Pareto point x_p.

In order to provide for the existence of balance points in the direction of preferential deviations, we consider the following condition which can be verified directly for a problem in question (and always holds if the balance set coincides with the balance space).

(C1) For a given $p \in P$ there exists a balance point in direction l.

2. APPLICATION OF THE KARUSH-KUHN-TUCKER THEOREM

Let us introduce the following scalar problem whose decision variables are $\tau \in \mathbb{R}_+$ and $x \in \Omega$:

$$\text{Min} \quad \tau \begin{cases} x \in \Omega, \\ g(x,p) \leq 0, \\ f(x,p) - \tau l \leq J(p), \ \tau \geq 0. \end{cases} \tag{3}$$

Due to A1, a finite value $J(p)$ exists, yielding the unique ideal point for (1) as the collection of partial minima for problems in (2). Hence, for sufficiently large $\tau > 0$, the feasible set $\{\tau, x\}$ in (3) is nonempty. Since this set is inf-compact (implied by A1, closedness of Ω and $\mathbb{R}_+$), the global solution $\tau^* = \min \tau(p)$, $x^* = x^*(p)$ of (3) always exists. This solution defines a point $b = \tau^* l$ of the balance space which, according to (C1), is expected to be a balance point. The corresponding $x^* = x^*(p)$ is then a Pareto solution for (1), see [1,3], and $f(x^*(p),p) = J(p) + \tau^* l$.

This allows us to compute the balance point and Pareto point associated with the direction of preferential deviations. In fact, at the first step the ideal point $J(p)$ can be computed by solving k scalar problems (2) and, later, once $J(p)$ is known, problem (3) leads to the balance point $\tau^* l$ and the Pareto point $x^*(p)$.

Let us include the constraint $x \in \Omega$ into the second line of (3) writing it in the same form and adding to the lines of vector inequality $g(x,p) \leq 0$.

Then the Lagrangian function of (3) is defined by

$$L(\tau, x, p, \mu, \nu) = \tau + \sum_{j=1}^{m} [\mu_j g_j(x,p)] + \sum_{i=1}^{k} \nu_i [f_i(x,p) - \tau l_i - J_i(p)], \tag{4}$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_m) \in \mathbb{R}^m$ and $\nu = (\nu_1, \nu_2, \dots, \nu_k) \in \mathbb{R}^k$. For brevity, we shall not write the parameter $p \in P$ and/or the vector $x \in \mathbb{R}^n$ in formulas below unless it is necessary for clarity of exposition.

ASSUMPTION A2. *Assume that Karush-Kuhn-Tucker (KKT) necessary conditions are applicable to problem (3) and that the global solution of (3) is unique and contained among the stationary points defined by KKT conditions.*

This assumption holds in many practical problems. Exact conditions (e.g., convexity) under which it is valid will be investigated elsewhere.

Denote by $x^*(p)$, $\tau^*(p) = \min \tau$ the unique global optimal solution of (3) corresponding to a fixed value of parameter $p \in P$. Then by the Karush-Kuhn-Tucker Theorem, at the point (x^*, τ^*)

[1] If $u, v \in \mathbb{R}^k$ with $u \leq v$, then $[u,v]$ denotes the set $\{x \in \mathbb{R}^k : u \leq x \leq v\}$, coordinate-wise.

the following conditions must hold:

$$\frac{\partial L}{\partial \tau} = 1 - \sum_{i=1}^{k} \nu_i^* l_i = 0, \tag{5}$$

$$\frac{\partial L}{\partial x_s} = \sum_{j=1}^{m} \mu_j^* \frac{\partial g_j}{\partial x_s} + \sum_{i=1}^{k} \nu_i^* \frac{\partial f_i}{\partial x_s} = 0, \qquad s = 1, \ldots, n, \tag{6}$$

$$\frac{\partial L}{\partial \mu_j} = g_j(x^*, p) \le 0, \qquad j = 1, \ldots, m, \tag{7}$$

$$\frac{\partial L}{\partial \nu_i} = f_i(x^*, p) - \tau^*(p) l_i - J_i(p) \le 0, \qquad i = 1, \ldots, k, \tag{8}$$

$$\mu_j^* g_j(x^*, p) = 0, \qquad j = 1, \ldots, m, \tag{9}$$

$$\nu_i^* \left[f_i(x^*, p) - \tau^*(p) l_i - J_i(p) \right] = 0, \qquad i = 1, \ldots, k, \tag{10}$$

$$\mu_j^* \ge 0, \quad \nu_i^* \ge 0, \qquad j = 1, \ldots, m, \quad i = 1, \ldots, k. \tag{11}$$

The existence of Lagrange multipliers μ_j^* and ν_i^* is included in Assumption A2. Equalities (5), (6), (9), (10) present $1+n+m+k$ equations for the same number of unknowns τ^*, x_s^*, μ_j^*, ν_i^*. By Assumption A2, these equations have one or more solutions satisfying (7), (8), (11) (and called *stationary* points) some of which deliver local minima for problem (3) and, by Assumption A2, one and only one of them yields the unique global $\min \tau = \tau^*(p)$. This solution (x^*, τ^*) defines (under Condition (C1)) a Pareto point $x^*(p)$, the balance point $b = \tau^*(p) l$ in direction $l \in \mathbb{R}^k$, and the actual values of cost functions $f_i(x^*(p), p) \le \tau^*(p) l_i + J_i(p)$ globally optimal in direction $l = (l_1, l_2, \ldots, l_k)$, thereby at least one of them will be given by the equality. To this solution there correspond certain values μ_j^*, ν_i^* of the Lagrange multipliers.

Nonlinear multiobjective problems have some interesting properties implied by the KKT conditions written for formulation (3).

2.1. The Balanced Problem Condition

If $\tau^*(p) = 0$, then by (8) we have

$$f_i(x^*, p) \le J_i(p), \qquad \forall\, i. \tag{12}$$

However, $J_i(p)$ are global minima for problems (2), i.e., partial minima yielding the ideal point $J(p) = (J_1, \ldots, J_k)$. Hence, it is impossible that $f_i(x^*, p) < J_i(p)$, so that we have

$$f_i(x^*, p) = J_i(p), \qquad \forall\, i, \text{ if } \tau^*(p) = 0. \tag{13}$$

It means that a single point $x^*(p)$ renders minimum for every cost function $f_i(x, p)$, thus, the problem is *balanced* (see [4, p. 138]) presenting, in fact, a scalar problem. To solve it, it is sufficient to solve (2) for just one index i, say, for $f_1(x)$, and then at its optimal point x^* compute all other $f_i(x^*)$ for $i = 2, \ldots, k$. Hence, in unbalanced problems (conflicting objectives), always $\tau^*(p) > 0$.

2.2. The Constraint Attraction Sufficient Condition

Usually, an unbalanced problem has its global optimal solution x^* in the interior of the feasible region, see examples in [1,2,4–6]. In this case, all $g_j(x^*, p) < 0$, thus, by (9), all $\mu_j^* = 0$. If some $\mu_j^* > 0$, then corresponding $g_j(x^*, p) = 0$ by (9). It means the absence of the cost functions repelling x^* from the constraint $g_j = 0$, hence, this constraint is attractive (profitable) with respect to the interests embodied in the vector cost function $f(x)$.

2.3. Sufficient Condition for a Balance Point in Terms of Lagrange Multipliers

If all $\nu_i^*(p) > 0$, then all brackets in (10) are zero, thus, noting that $b_i = \tau^*(p)l_i$, we have

$$f_i(x^*, p) = J_i(p) + b_i. \tag{14}$$

Since this is precisely the definition of a balance point, see [1,5] where the notation η_i is used instead of b_i, we conclude that, due to Condition (C1), equality (14) holds whenever the optimal vector $\tau^* l$ defines a balance point. However, the converse: if (14), then all $\nu_i^*(p) > 0$ is not implied and may not be true.

2.4. Necessary Condition of a Non-Pareto Point in Terms of Lagrange Multipliers

If $\tau^* l$ is not a balance point, then globally optimal x^*, τ^* still exist but Condition (C1) does not hold for that particular p in direction l and x^* is not a Pareto point, see [1,3]. In this case, at least one of the brackets in (10) is nonzero, so that the corresponding $\nu_i^*(p) = 0$. We see that the occurrence of a zero multiplier $\nu_i^*(p)$ indicates a possibility (but not the certainty) of violation of Condition (C1) for that p in direction l. If however, all $\nu_i^*(p) > 0$, then Condition (C1) must be satisfied for that particular $p \in P$.

3. ORTHOGONALITY IN MULTIOBJECTIVE PROGRAMS

For a fixed $p \in P$, the solution x^*, τ^* of system (5)–(11) depends on a choice of direction vector $l = (l_1, \dots, l_k)$. Let O be the origin of a reference system in $\mathbb{R}^k$ and $l(u)$ be a line described by the end of the variable radius-vector Ol (a hodograph of l). Suppose that $l(u)$ is C^1, i.e., once continuously differentiable, then the variation dl will correspond to infinitesimal tangent vector to the curve $l(u)$. To this variation $dl(u)$ will correspond the variations of all variables in (5)–(11) except for p and $J(p)$ which are constants. In particular, the vector $\nu^*(u)$ will describe a curve in the space $\{\nu^*\}$, and the point $b(u) = \tau^*(u)l(u)$ will follow a curve in the balance space $\{b\}$ associated with the problem.

THEOREM 1. *If $\nu^*(u)$ is C^1, then the tangent vectors $d\nu^*$ and dl are orthogonal in superimposed spaces $\{\nu^*\}, \{l\}$*

$$d\nu^* \cdot dl = 0. \tag{15}$$

PROOF. Equality (5) can be written in vector form as the scalar product

$$\nu^* l = 1. \tag{16}$$

With corresponding small variations, we have

$$(\nu^* + d\nu^*)(l + dl) = \nu^* l + \nu^* dl + d\nu^* l + d\nu^* dl = 1. \tag{17}$$

On the other hand, differentiating (16) yields

$$d(\nu^* l) = d\nu^* l + \nu^* dl = 0. \tag{18}$$

Comparing (16)–(18) proves (15). ∎

THEOREM 2. *If $\nu^*(u)$ and $\tau^*(u)$ are both C^1, then so is $b(u)$, and for unbalanced problems the tangent vectors $d\nu^*$ and db are orthogonal in superimposed spaces $\{\nu^*\}, \{b\}$*

$$d\nu^* \cdot db = 0. \tag{19}$$

PROOF. The curve $l(u)$ is C^1 by the choice of initial variations of direction l. If such is also $\tau^*(u)$, then $b(u) = \tau^*(u)l(u)$ is C^1. For unbalanced problems, we have $\tau^*(u) > 0$, so that multiplying (16) by $\tau^*(u)$ we get

$$\nu^* \cdot b = \tau^*. \tag{20}$$

With corresponding small variations, we have

$$(\nu^* + d\nu^*)(b + db) = \nu^* b + \nu^* db + d\nu^* b + d\nu^* db = \tau^* + d\tau^*. \tag{21}$$

On the other hand, differentiating (20) yields

$$d(\nu^* b) = d\nu^* b + \nu^* db = d\tau^*. \tag{22}$$

Comparing (20)–(22) proves (19). ∎

COROLLARY 2.1. *If $b(u)$ is a straight line, then such is also $\nu^*(u)$ and vice versa, and both lines are orthogonal in superimposed spaces $\{b\}, \{\nu^*\}$.*

COROLLARY 2.2. *Since the rotating ray Ol sweeps the whole balance space $\{b\}$, for some family of curves $l(u)$ it sweeps the entire balance set, $\tau^* l(u) = \tilde{b}(u) \in B \subseteq \{b\}, \cup \tilde{b}(u) = B$, so the corresponding lines $\{\nu^*(u)\}$ form a family orthogonal to the balance set in the sense (19).*

Suppose now that the vector-parameter $p \in P \subseteq \mathbb{R}^q$ varies giving rise to curves $\tau^*(v)$, $\nu_i^*(v)$, $i = 1, \dots, k$, where v is a scalar parameter of a curve (a hodograph) described by the radius-vector $p(v)$.

THEOREM 3. *If $l =$ const and $\tau^*(v)$, $\nu^*(v)$ are C^1, then*

$$d\nu^* \cdot l = 0, \qquad d\nu^* \cdot db = 0. \tag{23}$$

PROOF. It is similar to those of Theorems 1 and 2. ∎

THEOREM 4. *If l, p both vary and $l(u)$, $\tau^*(u,v)$, $\nu^*(u,v)$ are all C^1 curves, then we have, to the second order*

$$d\nu^* d\tau^* l + \nu^* d\tau^* dl + d\nu^* \tau^* dl = 0. \tag{24}$$

PROOF. Subtracting $\nu^* \tau^* l = \tau^*$ and $d(\nu^* \tau^* l) = d\tau^*$ from the equality

$$(\nu^* + d\nu^*)(\tau^* + d\tau^*)(l + dl) = \tau^* + d\tau^* \tag{25}$$

and dropping the term $d\nu^* d\tau^* dl$, we get (24). ∎

REMARK. If $p =$ const, then $\tau^* =$ const > 0 and (24) reduces to (15),(19). If $l =$ const, then (24) reduces to (23). It is worth noting that b in (19) to (23) is not necessarily a balance point.

All four theorems are valid also for piecewise smooth curves if one takes the corresponding one-side differentials.

4. CONCLUSIONS

Several interesting properties of nonlinear multiobjective problems implied by the Karush-Kuhn-Tucker necessary conditions are investigated. It is shown that trajectories of Lagrange multipliers corresponding to the components of the vector cost function are orthogonal to the corresponding trajectories of vector deviations in the balance space (to the balance set for Pareto solutions).

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